52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Asymptotic Consensus Solutions in Non-Linear Delayed Distributed Algorithms Under Deterministic & Stochastic Perturbations

Christoforos Somarakis and John S. Baras

Abstract— **We consider a multi-agent non-linear delayed model which sustains consensus type of solutions. We use fixed point theory arguments to establish sufficient conditions for existence and uniqueness of solutions that converge exponentially fast to a common value with prescribed rate. The conditions depend on the communication topology, the non-linearity of the model as well as the delay in the propagation of information. Furthermore we test the robustness of our results in the presence of independent time varying perturbations both deterministic and stochastic.**

I. INTRODUCTION

Consensus problems are important in different fields of Applied Science, from distributed computing and optimization in the Control Community, to flocking models in Applied Mathematics and Physics (see [1], [2], [3], [4], [5] and references therein). The fundamental mechanism for this collective behavior is a distributed agreement protocol algorithm under which the autonomous agents exchange their state information locally. Let $[N] = \{1, \dots, N\}$ be a finite collection of agents. In continuous time, the rate of change of the state $x_i(t)$ of agent $i \in [N]$, is a weighted zero sum average of the states collected locally. The resulting system of differential equations is of the form

$$\dot{x}_i = \sum_j a_{ij}(x_j - x_i). \tag{1}$$

It is very well known that these types of dynamics typically sustain solutions which asymptotically converge to a common value, known as the consensus value. That is, there exists z such that $|x_i(t) - z| \to 0$ as $t \to \infty, \forall i \in [N]$. The stability analysis of these systems are in-principle studied through a Lyapunov-based argument.

A. Delayed Models

Two crucial hypotheses in the analysis of the consensus algorithms are: 1) The dynamics are linear and 2) The distribution of information is synchronous. These two assumptions are in reality overly simplistic. To the best of our knowledge we mention the following related works that tried to address these issues:

In [2], [6] the authors consider a discrete time version of (1) with multiple, time varying delays. Their strategy of attacking the problem is to extend the state space by adding artificial agents which play no actual role in the dynamics other than transmitting a pre-described delayed version of an agent's state. This method, although applicable in discrete time, it is unclear how it would work in a continuous time system, unless the latter one is solved numerically. For continuous time systems, Olfati-Saber et al. [7] use frequency domain methods to prove asymptotic consensus in a linear time invariant consensus algorithm where the delay is uniform for all agents and the dynamics were computed in an asynchronous manner (i.e. both x_i and x_j where delayed values in (1)). Finally, in [8] the authors discuss the convergence properties of the non-linear delayed algorithm

$$\dot{x}_i = \sum_j A_{ij} f_{ij}(x_j(t - \tau_j^i) - x_i(t)). \tag{2}$$

By imposing passivity assumptions on the f_{ij}'s, they apply Invariance Principles to derive delay-independent asymptotic consensus. The main setback of this approach, also noted by the authors, is that nothing can be said about either the consensus point or the rate of convergence to it.

1) Motivation and Contribution: Our paper considers a non-linear variation of the consensus model with structural delays in the communication between agents, in the presence of uncertainty signals which play the role of noise. Given $N < \infty$ autonomous agents the model we will consider is:

$$\dot{x}_i(t) = \sum_j g_{ij}\big(x_j(t - \tau_i^j)\big) - g_{ij}\big(x_i(t)\big) + q_i(t) \tag{MDL}$$

where g_{ij} are the non-linear interactions so that each agent i has a different structural way of analyzing the signal from its neighbor $j \in N_i$. Moreover, each agent is equipped with a function q_i which models the uncertainty in the calculated signal. The properties on g_{ij} and q_i are to be defined in the following.

This model significantly differs from the ones known in the literature in many ways, the most important of which is that we will not assume any monotonicity assumptions on g_{ij}'s. Our goal is to derive sufficient conditions on the system parameters so that exponential type of convergence to a consensus value is guaranteed. Unlike the majority of the works in literature we will use a fixed point theory argument in complete metric spaces.

2) Organization of the paper: This paper is organized as follows. In Section (II), we will provide the preliminary mathematical tools needed for the analysis of (MDL). In Section (III), we will exercise the Fixed Point Theory method to derive results in the case of a uniform positive delay

This work was partially supported by US Army Research Office MURI grant W911NF-08-1-0238, by US Air Force Office of Scientific Research MURI grant FA9550-09-1-0538, and by National Science Foundation (NSF) grant CNS-1035655

Christoforos Somarakis is with the Institute for Systems Research, University Of Maryland, College Park csomarak@umd.edu

John S. Baras is with the Institute for Systems Research, University Of Maryland, College Park baras@umd.edu

978-1-4673-5716-6/13/$31.00 ©2013 IEEE

among all agents and no disturbances. Next, in Section (IV), we elaborate on the results obtained so far to establish sufficient conditions for the consensus problems with deterministic disturbances. Then in Section (V) we switch to non-deterministic disturbances and arrive at a stochastic version of (MDL). We will see that the induced statistical regularity strengthens the results of Section (IV) by providing, in addition to sufficient, necessary conditions.

In Section (VI) we summarize our results; A large part is devoted to the pros and cons of the fixed point theory approach. We conclude the paper by discussing important extensions of our model and the way to deal with them within the theoretical framework of this work.

II. PRELIMINARIES

A. Linear Spaces and Norms

The dynamics evolve in the N-dimensional Euclidean space, $\mathbb{R}^N$, which is endowed with the norm $||\mathbf{x}|| = \sum_{i=1}^N |x_i|$ and the corresponding induced matrix norm. By $\mathbf{1}$ we understand the N dimensional vector of all ones. The subspace of interest is

$$\Delta := \{\mathbf{y} \in \mathbb{R}^N : \mathbf{1}z, z \in \mathbb{R}\}$$

and it is called the consensus subspace. $C[I_1, I_2]$ denotes the set of continuous functions defined in I_1 and taking values in I_2. So long as I_1 is unbounded we will assume bounded elements in C, with respect to the the supremum norm, $|\phi| := \sup_t ||\phi(t)||$ which in any case will be our standard metric for these spaces. By $L^p_{[a,b]}$ we understand the space of p-integrable functions in $[a, b]$ and by $Lip_{\mathcal{J}}[a, b]$ we understand the space of globally Lipschitz continuous functions defined in $[a, b]$ with Lipschitz constant $\mathcal{J}$.

B. Graph Theory & Agreement Dynamics

The communication system is modeled through the mathematical object of a weighted graph, $G = ([N], E, W)$ with $[N] = \{1, \ldots, N\}$ denoting the set of vertices (agents), E a family of couples $(i, j) : i, j \in [N]$, each member of which denotes an established communication from j to i. Hence each agent i has a set of agents adjacent to him denoted by N_i (the cardinality of which is $|N_i|$). We will also use the notation $\bar{N} := \sum_i |N_i|$, $N_M := \max_i |N_i|$, $\sum_i := \sum_{i=1}^N$ and $\sum_{i,j} := \sum_{i=1}^N \sum_{j \in N_i}$. In this paper, we consider graphs with symmetric weights (simply connected), so that information is eventually propagated freely throughout the population. Graphs are adequately represented by matrices and this approach lies in the field of Algebraic Graph Theory (see for example [5], [9], [10]). The most important matrix for this work is the Laplacian matrix, L, the properties of which are very well known in the literature. Given a graph G and it's weighted Laplacian matrix, L, the initial value problem

$$\dot{\mathbf{y}} = -L\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}_0 \tag{3}$$

is in fact the vector form of (1) and it is called a system of linear agreement dynamics. It is well known that for a connected graph all solutions converge exponentially fast to a common constant value $\mathbf{1}^T\mathbf{y}_0$. The rate of convergence is exponential with rate equal to the second smallest eigenvalue of L, which in the case of G being simply connected is positive (see [5]). The following Lemma, proved in [4], states two useful bounds on the rate of convergence:

Lemma 1: Given the weighted Laplacian, L, of a simply connected symmetric graph, together with its spectrum λ_i, the following bounds hold:

$$||e^{-L(t-s)}L|| \leq \sqrt{N}\lambda_N e^{-\lambda_2(t-s)}$$
$$\left|\left|e^{-L(t-s)} - \frac{\mathbf{1}\mathbf{1}^T}{N}\right|\right| \leq \sqrt{N}e^{-\lambda_2(t-s)}$$

C. Fixed Point Theory

A complete metric space is a pair $(\mathcal{S}, \rho)$ where $\mathcal{S}$ is a non-empty set and $\rho : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ is a function satisfying the properties of a metric; with the property that every Cauchy sequence in $(\mathcal{S}, \rho)$ has a limit in that $\mathcal{S}$. Perhaps the most fundamental result in Fixed Point Theory is the following Principle.

Theorem 1 (Contraction Mapping Principle): Let $(\mathcal{S}, \rho)$ be a complete metric space and let $P : \mathcal{S} \to \mathcal{S}$. If there is a constant $\alpha \in [0, 1)$ such that for each pair $\phi_1, \phi_2 \in \mathcal{S}$ we have

$$\rho(P\phi_1, P\phi_2) \leq \alpha\rho(\phi_1, \phi_2) \tag{4}$$

then there is a unique point $\phi \in \mathcal{S}$, such that $P\phi = \phi$.
The proof of this theorem can be found in any advanced analysis or ODE textbook (see for example [11]). In the following, the symbol α will be abused in the phrase "*there exists* $\alpha \in [0, 1)$ " for characterizing different and uncorrelated quantities.

D. Stochastic Processes

The standard set-up for a continuous time N-dimensional stochastic process

$$\mathbf{X} = \{\mathbf{X}(t) = (X_1(t), \ldots, X_N(t)) : t \geq 0\}$$

is the complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where Ω is the non-empty set of all possible outcomes, $\mathcal{F}$ is the collection of all $\mathbb{P}$-measurable events and $\mathcal{F}_t$ is the natural filtration generated by the standard N-dimensional Brownian motion, denoted as $\mathbf{B} = \{\mathbf{B}(t); t \geq 0\}$. We recall the definition of the *Itô* integral defined for any $\mathcal{F}_t$-adapted process, X and fixed $t > 0$

$$I_X(t) = \int_0^t X(s)dB(s)$$

It is reminded that if, in addition, X is a continuous process then I_X is a local martingale, whereas if $X(s)$ is a continuous deterministic function (i.e. $\mathcal{F}(0)$-adapted and in $C([0, \infty], \mathbb{R})$ then the local martingale is a normally distributed random variable with zero mean and $\int_0^t X^2(s)ds$ variance (see [12]).

III. The Unperturbed Delayed Model

For a population of $N < \infty$ agents we consider the initial value problem:

$$\begin{aligned} \dot{x}_i(t) &= \sum_{j \in N_i} g_{ij}(x_j(t-\tau)) - g_{ij}(x_i(t)), \quad t > 0 \\ x_i(t) &= \phi_i(t), \qquad -\tau \le t \le 0 \end{aligned} \tag{5}$$

This system is a special case of (MDL) with $\tau > 0$ a uniform delay in all the received signals and $q_i \equiv 0$. The vector valued function $\boldsymbol{\phi}(t) := (\phi_1(t), \ldots, \phi_N(t))$ models the initial data.

A. Assumptions, Main Result and Analysis

The set of assumptions to be used is the following:

Assumption 1: $\boldsymbol{\phi} \in C([-\tau, 0], \mathbb{R}^N)$

Assumption 2: The communication graph is static and the corresponding topological graph is simply connected.

Assumption 3: $\forall i, j \in N_i : \; g_{ij}(\cdot) \in Lip_{\mathcal{L}}(\mathbb{R}, \mathbb{R})$ are symmetric functions with $|g_{ij}| \le C$.

Define the vector valued function $\mathbf{f}(\mathbf{x})$ with elements $f_i(\mathbf{x}) = -\sum_{j \in N_i} a_{ij}(x_j - x_i) + \sum_{j \in N_i} \big(g_{ij}(x_j) - g_{ij}(x_i)\big)$ for some positive numbers a_{ij} where $j \in N_i$.

Assumption 4: $\forall j \in N_i \; \exists \; a_{ij} > 0$ such that $\mathbf{f} \in Lip_{\mathcal{K}}(\mathbb{R}^N, \mathbb{R}^N)$.

Assumption 5: $\exists \alpha \in [0, 1)$ such that $\frac{\tau \bar{N} \mathcal{L}}{N} \le \alpha$.

a) Metric Spaces: Fix $d > 0$, $\tau > 0$, $|z| < \infty$, $\boldsymbol{\phi}$ as in Assumption (1). We will look for solutions in the metric space $(\mathcal{M}, \rho_d)$ with

$$\begin{aligned} \mathcal{M} = \mathcal{M}_{d,z,\tau,\boldsymbol{\phi}} = \{ &\mathbf{y} \in C([-\tau, \infty), \mathbb{R}^N) : \\ &y_i = \phi_i|_{[-\tau,0]}^{i \in [N]}, \sup_{t \ge -\tau} e^{dt} ||\mathbf{y}(t) - \mathbf{1}z|| < \infty \} \end{aligned}$$

and the function:

$$\rho_d(\mathbf{y}_1, \mathbf{y}_2) = \sup_{t \ge 0} e^{dt} ||\mathbf{y}_1(t) - \mathbf{y}_2(t)||.$$

Lemma 2: The metric space $(\mathcal{M}, \rho_d)$ is complete.

Proof: [Sketch] The proof is a straightforward application of the definition of a complete metric space. See [4]. ■

$\mathcal{M}$ is the space of bounded continuous functions that coincide with ϕ in $[-\tau, 0]$ and converge exponentially fast with rate d to a common finite point $\mathbf{1}z \in \Delta$. Now, consider the graph $G = \{[N], E, W\}$ with edge weights a_{ij}. Then for the spectrum of the Laplacian of G, we have $\lambda_2 > 0$.

Assumption 6: We choose d such that $0 < d < \lambda_2$.

Assumption 7: There exists $\alpha \in [0, 1)$ such that

$$\frac{\sqrt{N}\mathcal{K}}{\lambda_2 - d} + \frac{e^{d\tau} - 1}{d} N_M \mathcal{L} \Big(1 + \frac{\sqrt{N}\lambda_N}{\lambda_2 - d} \Big) \le \alpha$$

The notation is explained in Section (II).

1) Statement of the Main Result:

Theorem 2: Consider the initial value problem (5). Under Assumptions (1)-(7), all solutions tend to a common value exponentially fast with rate d.

Proof: The proof is an application of Theorem 1. We will define a solution operator and prove that it is a contraction in $(\mathcal{M}, \rho)$. For this we will need Propositions 1 and 2 in the following. ■

Before proceeding in the analysis we make an important comment about the limiting value.

2) Consensus Point: Under Assumption 2, we sum over all $i \in [N]$ and obtain:

$$\begin{aligned} &\sum_i \dot{x}_i = -\frac{d}{dt} \sum_{i,j} \int_{t-\tau}^{t} g_{ij}(x_j(s)) ds \Rightarrow \\ &\sum_i x_i(t) = \sum_i \phi_i(0) + \sum_{i,j} \Big[\int_{-\tau}^{0} g_{ij}(\phi_j(s)) ds - \\ &\qquad\qquad - \int_{t-\tau}^{t} g_{ij}(x_j(s)) ds \Big] \end{aligned}$$

If $x_i(t) \to z$ then it is necessary that z satisfies

$$N \cdot z = \sum_i \phi_i(0) + \sum_{i,j} \int_{-\tau}^{0} g_{ij}(\phi_j(s)) ds - \tau \sum_{i,j} g_{ij}(z) \tag{6}$$

Since our system is non-linear there is no guarantee that there is a $z \in \mathbb{R}$ satisfying the above non-linear equation. We need to derive sufficient conditions so that z is the unique solution of this non-linear equation.

Lemma 3: Under Assumption (5) there exists a unique $z = z(N, g_{ij}, \phi, \tau)$ satisfying Eq. (6).

Proof: For fixed ϕ, τ, g_{ij}, eq. (6) can be written in the form

$$J_{\phi,\tau,g_{ij},N}(z) = C_1(\phi, N, g_{ij}) + C_2(\phi, N, g_{ij}, z).$$

Then in the complete metric space $(\mathbb{R}, |\cdot|)$ for $z_1, z_2 \in \mathbb{R}$

$$|J_{\phi,\tau,g_{ij},N}(z_1) - J_{\phi,\tau,g_{ij},N}(z_2)| \le \frac{\tau \bar{N} \mathcal{L}}{N} |z_1 - z_2|$$

which is a contraction. Then Theorem 1 guarantees a unique solution of Eq. (6). ■

3) Analysis: Consider the Laplacian matrix of the associated weighted graph with weights a_{ij}. Then we write (5) as follows

$$\dot{\mathbf{x}} = -L\mathbf{x} + \mathbf{f}(\mathbf{x}) - \frac{d}{dt} \mathbf{g}(\mathbf{x}) \tag{7}$$

where $\mathbf{f}$ is as defined in Assumption 4 and $\mathbf{g}$ is a vector valued function with elements $g_i(\mathbf{x}) = \sum_{j \in N_i} \int_{t-\tau}^{t} g_{ij}(x_j(s)) ds$. Note that due to symmetry $\mathbf{1}^T \mathbf{f} = 0$ and for $z \in \mathbb{R}$, $\frac{d}{dt}\mathbf{g}(\mathbf{1}z) = 0$. These elementary properties will come at hand soon. We brought (5) in a form appropriate enough to define the solution operator. Indeed using the Variation of Constants formula:

$$\mathbf{x}(t) = e^{-Lt} \boldsymbol{\phi}(0) + \int_0^t e^{-L(t-s)} \Big[\mathbf{f}(\mathbf{x}(s)) - \frac{d}{ds} \mathbf{g}(\mathbf{x}(s)) \Big] ds$$

We define the operator $P:\mathcal{M}\to\mathcal{B}$ as follows:

$$(P\mathbf{y})(t)=\begin{cases}\boldsymbol{\phi}(t), & -\tau\le t\le 0\\ e^{-Lt}\boldsymbol{\phi}(0)+\int_0^t e^{-L(t-s)}\mathbf{f}(\mathbf{y}(s))ds- & \\ -\int_0^t e^{-L(t-s)}\frac{d}{ds}\mathbf{g}(\mathbf{y}(s))ds, & t>0\end{cases}$$

Proposition 1: If z satisfies Eq. (6) and $d<\lambda_2$, then $P:\mathcal{M}\to\mathcal{M}$

Proof: It suffices to check that the function $(P\mathbf{y})$ defines a function that is a member of $\mathcal{M}$ by verifying the properties of this space, one by one. The boundedness and the identity to the given initial data are readily derived by the definition of the operator.

Now, as $t\to\infty$, for any $\mathbf{x}\in\mathcal{M}$, $e^{-Lt}\phi(0)\to\frac{1}{N}\mathbf{1}^T\phi(0)$; the second term vanishes to zero since by the orthogonality of $\mathbf{f}$ to the consensus space it is equal to

$$\int_0^t\left(e^{-L(t-s)}-\mathbf{1}\mathbf{1}^T\frac{1}{N}\right)\mathbf{f}(\mathbf{x}(s))ds$$

and this is the convolution of an L^1 function with a function that goes to zero (that is, $\mathbf{f}$). The last term vanishes likewise if one adds and substracts to the last term the integrand $\frac{\mathbf{1}\mathbf{1}^T}{N}\frac{d}{ds}(\mathbf{g}(\mathbf{x}(s))-\mathbf{g}(\mathbf{1}z))$. The remaining term $\mathbf{1}\frac{1}{N}\int_0^t\frac{d}{ds}\mathbf{1}^T(\mathbf{g}(\mathbf{x}(s))-\mathbf{g}(\mathbf{1}z))ds$ belongs to Δ and hence is of the form $\mathbf{1}b(t)$ where

$$\begin{aligned}b(t):=&\frac{1}{N}\sum_{i,j}\int_{t-\tau}^t g_{ij}(x_j(s))-g_{ij}(z)ds\\&+\frac{1}{N}\sum_{i,j}\int_{-\tau}^0 g_{ij}(\phi_j(s))-g_{ij}(z)ds.\end{aligned}$$

which converges to $\frac{1}{N}\sum_{i,j}\int_{-\tau}^0 g_{ij}(\phi_j(s))-g_{ij}(z)ds$ as $t\to\infty$. Then, the whole expression converges to a non-linear expression of z, which is identical to (6). By Lemma 3, this expression has a unique fixed point, proving that the operator introduces a function which eventually converges to this particular point in Δ on condition of the assumptions of Lemma 3.

Next, we have to prove that this happens with the same rate of convergence, i.e. $\sup_t e^{dt}||(P\mathbf{x})(t)-\mathbf{1}z||<\infty$. This condition is readily derived from the preliminary results on the rate of convergence of $e^{-L(t-s)}-\frac{\mathbf{1}\mathbf{1}^T}{N}$ to zero (that is, exponential with rate $\lambda_2>0$) and the fact that all the convoluted quantities are either bounded, or vanish at the same rate d as members of $\mathcal{M}$. So, Assumption 6 suffices. ■

Proposition 2: Under Assumption 7, $P:\mathcal{M}\to\mathcal{M}$ is a contraction in $(\mathcal{M},\rho_d)$.

Proof: For any $\mathbf{x}_1,\mathbf{x}_2\in\mathcal{M}$ and $t>0$, applying integration by parts and after considerable care we arrive at:

$$\begin{aligned}&||(P\mathbf{x}_1)(t)-(P\mathbf{x}_2)(t)||\le\\&\le\int_0^t\left|\left|e^{-L(t-s)}-\frac{\mathbf{1}\mathbf{1}^T}{N}\right|\right|\cdot||\mathbf{f}(\mathbf{x}_1(s))-\mathbf{f}(\mathbf{x}_2(s))||ds\\&+\int_0^t||e^{-L(t-s)}L[\mathbf{g}(\mathbf{x}_1(s)))-\mathbf{g}(\mathbf{x}_2(s))]||ds\\&+||\mathbf{g}(\mathbf{x}_1(t)))-\mathbf{g}(\mathbf{x}_2(t))||\\&\le\sqrt{N}\mathcal{K}\int_0^t e^{-\lambda_2(t-s)}||\mathbf{x}_1(s)-\mathbf{x}_2(s)||ds\\&+\sqrt{N}\lambda_N N_M\mathcal{L}\int_0^t e^{-\lambda_2(t-s)}\int_{s-\tau}^s||\mathbf{x}_1(w)-\mathbf{x}_2(w)||dwds\\&+N_M\mathcal{L}\int_{t-\tau}^t||\mathbf{x}_1(s)-\mathbf{x}_2(s)||ds\\&\le\sqrt{N}\mathcal{K}\frac{e^{-dt}-e^{-\lambda_2 t}}{\lambda_2-d}\cdot\rho_d(\mathbf{x}_1,\mathbf{x}_2)\\&+\sqrt{N}\lambda_N N_M\mathcal{L}\frac{e^{d\tau}-1}{d}\frac{e^{-dt}-e^{-\lambda_2 t}}{\lambda_2-d}\cdot\rho_d(\mathbf{x}_1,\mathbf{x}_2)\\&+N_M\mathcal{L}\frac{e^{d\tau}-1}{d}e^{-dt}\cdot\rho_d(\mathbf{x}_1,\mathbf{x}_2).\end{aligned}$$

So finally

$$\begin{aligned}&\rho_d(P\mathbf{x}_1,P\mathbf{x}_2)=\sup_{t\ge0}e^{dt}||P\mathbf{x}_1(t)-P\mathbf{x}_2(t)||\le\\&\left(\frac{\sqrt{N}\mathcal{K}}{\lambda_2-d}+\frac{e^{d\tau}-1}{d}N_M\mathcal{L}\big(1+\frac{\sqrt{N}\lambda_N}{\lambda_2-d}\big)\right)\rho_d(\mathbf{x}_1,\mathbf{x}_2)\end{aligned}$$

making P a contraction under Assumption (7). ■

IV. Deterministic Perturbations

In this section we make a minor, yet important for what follows, extension of (5). We consider, the perturbed system

$$\dot{x}_i(t)=\sum_{j\in N_i}g_{ij}\big(x_j(t-\tau)\big)-g_{ij}\big(x_i(t)\big)+q_i(t).\quad(8)$$

where $\mathbf{q}(t)=(q_1(t),\ldots,g_N(t))\in C^0([0,\infty),R^N)$ models an external (i.e. state independent) deterministic perturbation. One would suspect that (5) and (8) exhibit qualitatively equal asymptotic behaviour, so long as $\mathbf{q}(t)$ vanishes sufficiently fast. Using the same techniques as those in the proof of Theorem 2, we obtain the following result

Theorem 3: If there is an $s>0$ such that $\sup_t e^{st}||\mathbf{1}^T\mathbf{q}(t)||<\infty$ then (5) converges to a constant value exponentially fast with rate $d<\min\{\lambda_2,s\}$.

Proof: [Sketch] The proof of the above result, although it cannot be stated as a corollary of Theorem 2, it, nonetheless, bears no significant difference from the proof of Theorem 2 and it will thus be omitted due to space limitations. The only differences are in the operator solution and the consensus point, which in this case needs the full orbital information $\mathbf{q}$ since the system in non-autonomous and cannot only depend on the initial data. ■

V. Stochastic Perturbations

We consider a special, yet important, case of perturbations and that is the case where the dynamics of each agent i are perturbed with an external standard Brownian motion $B_i(t)$ independent of the state and independent from the perturbations of other agents. We work on a complete filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ as defined in Section (II). For $i=1,\ldots,N$ the system of stochastic functional equations is written for $t>0$ as follows :

$$dX_i(t)=\sum_{j\in N_i}g_{ij}(X_j(t-\tau))-g_{ij}(X_i(t))dt+\sigma_i(t)dB_i(t)\quad(9)$$

where $X_i(t,\omega)=\phi_i(t)$ for $t\in[-\tau,0]$ and for almost all $\omega\in\Omega$.

The solutions of (9) are $\mathcal{F}_t$-adapted processes that are generated by $\mathbf{B}(t) = (B_1(t), \ldots, B_N(t))$ and the consensus point is a finite $\mathcal{F}_\infty$-measurable random variable to be further characterized. It is the goal of this section to investigate under which assumptions the solutions of (9) enjoy the same asymptotic behaviour of (8) and in what sense.

a) Existence / Uniqueness: We should make a small comment and stress that unlike the deterministic case where the Fixed Point Argument covered the issue of existence in the large and uniqueness for the solutions of (5) and (8), in this case one needs to a priori guarantee these fundamental properties for the IVP (9). On condition that, Assumptions (3) and (8)-1 below, hold; the IVP (9) admits a unique t-continuous $\mathcal{F}_t$-adapted process $\mathbf{X}_t = (X_1(t), \ldots, X_N(t))$. This process is unique in the sense that any other process $\tilde{\mathbf{X}}$ is a.s. indistinguishable from $\mathbf{X}$ i.e.

$$\mathbb{P}\big[\{\omega \in \Omega : \mathbf{X}_t(\omega) = \tilde{\mathbf{X}}_t(\omega), \forall t \geq -\tau\}\big] = 1$$

For more on the theory of functional SDEs see [14], [12], [11].

A. Assumptions and Main Result

Together with the Assumptions already stated in previous sections we need the following one:

Assumption 8: The deterministic perturbation functions $\sigma_i|_{i\in[N]}$ satisfy the following properties:

1. $\sigma_i \in C([0,\infty], \mathbb{R}^N)$
2. $\mathbb{P}\big[\{\omega \in \Omega : \sup_t e^{dt}\big(\int_0^t \sigma_i(s)dB_i(s)\big)(\omega) < \infty\}\big] = 1$

This assumption describes the asymptotic behaviour of σ_i's needed to guarantee almost sure exponential convergence for (9). The theorem below summarizes the main result:

Theorem 4: Under Assumptions (1)-(8), consider the unique t-continuous $\mathcal{F}_t$-adapted process $\mathbf{X}_t$ which satisfies (9). Then the two statements below are equivalent:

a. There exists an $\mathcal{F}_\infty$-measurable and a.s. finite random variable z which satisfies

$$N \cdot z = -\tau \sum_{i,j} g_{ij}(z) + \sum_{i,j} \int_{-\tau}^{0} g_{ij}(\phi_j(s))ds + \\ + \sum_i \phi_i(0) + \sum_i \int_0^\infty \sigma_i(s)dB^i(s) \tag{4.a}$$

such that

$$\lim_{t\to\infty} \mathbf{X}_t = \mathbf{1}z \quad \text{a.s.}$$

b. $\sigma_i \in L^2([0,\infty), \mathbb{R}), \quad \forall i \in [N]$.

1) Consensus point: Similarly to the case of the deterministic framework where z is a solution of a non-linear equation, in the case of Brownian motion stochastic perturbations one would be interested in characterizing the distribution of z.

Proposition 3: If z is the random variable defined by (4.a), then it has the same distribution as the random variable $T^{-1}(U)$, where $T \in C(\mathbb{R}, \mathbb{R})$ is defined by $T(x) = Nx + \tau \sum_{i,j} g_{ij}(x)$ and U is the normal random variable $\mathcal{N}(\sum_i \phi_i(0) + \sum_{i,j} \int_{-\tau}^0 g_{ij}(\phi_j(s))ds, \sum_i \int_0^\infty \sigma_i^2(s)ds)$

Proof: From Theorem 4, we have that $\sigma_i \in L^2$. In view of the independence of the stochastic processes $B_i(t)$ it follows that $\int_0^\infty \sigma_i(s)dB_i(s)$ is a normal random variable with zero mean and variance $\int_0^\infty \sigma_i^2(s)ds < \infty$. The expression in (4.a) implies that the random variable

$$U := \sum_i \phi_i(0) + \sum_{i,j} \int_{-\tau}^0 g_{ij}(\phi_j(s))ds + \sum_{i,j} \int_0^\infty \sigma_i(s)dB_i(s)$$

is normally distributed with mean $\sum_i \phi_i(0) + \sum_{i,j} \int_{-\tau}^0 g_{ij}(\phi_j(s))ds$ and variance $\sum_i \int_0^\infty \sigma_i^2(s)ds$. Note that $T(z) = U$ and the result of the Lemma follows from the fact that T is an increasing function of x. For $x_1 \geq x_2$

$$T(x_1) - T(x_2) = x_1 - x_2 + \sum_{i,j} \big(g_{ij}(x_1) - g_{ij}(x_2)\big) \\ \geq x_1 - x_2 - \tau \bar{N}\mathcal{L}(x_1 - x_2) > 0$$

by Assumption (6). So the distribution of z is:

$$\mathbb{P}[z \leq x] = \mathbb{P}[T^{-1}(U) \leq x] = \mathbb{P}[U \leq T(x)] = \\ = \Phi\left(\frac{T(x) - \sum_i \phi_i(0) - \sum_{i,j} \int_{-\tau}^0 g_{ij}(\phi_j(s))ds}{\sqrt{\int_0^\infty \sum_i \sigma_i^2(s)ds}}\right)$$

where $\Phi(\cdot)$ is the distribution of the standard normal random variable. ∎

Proof: [Theorem 4]

[b→a] Since σ_i are L^2 functions, the Martingale Convergence Theorem [14] assures that $\lim_t \int_0^t \sigma_i(s)dB^i(s)$ exists a.s. and it is an a.s. finite random variable. Also from the analysis of the previous sections the solution formula of the process which obeys (9) is

$$\mathbf{X}_t = e^{-Lt}\mathbf{X}_0 + \int_0^t e^{-L(t-s)}\big(\mathbf{f}(\mathbf{X}_s) - \frac{d}{ds}\mathbf{g}(\mathbf{X}_s)\big)ds \\ + \sum_i \int_0^t e^{-L(t-s)}\hat{\boldsymbol{\sigma}}^i(s)dB^i(s) \tag{10}$$

where $\hat{\boldsymbol{\sigma}}^i(s) = (0, \ldots, \sigma_i(s), 0, \ldots, 0)^T$. For the expression of solutions see also [12] and note that $\frac{d}{ds}\mathbf{g}$ is well-defined even for $\mathbf{X}_s$ being differentiable almost nowhere. Consider the event $A \in \mathcal{F}$

$$A_a := \{\omega \in \Omega : \lim_t \big(\int_0^t \sigma_i(s)dB^i(s)\big)(\omega) \text{ exist } \forall i \in [N], \\ \text{and } \mathbf{X}_t(\omega) \text{ is the solution of (9)}\}.$$

Then $\mathbb{P}[A_a] = 1$ and for any fixed $\omega \in A_a$ the realisation $\mathbf{X}_\omega(t)$ is expressed by (10) and it bears no difference from Theorem 3, if one replaces the condition for $\mathbf{q}$ in Theorem 3 with Assumption (8)-2 . Also for any fixed ω there is a fixed $z = z_\omega$ by the techniques of Section III. In both cases the corresponding metric spaces upon which the fixed point arguments are established are parametrized by ω. Moreover, the fact that the perturbations are taken to be independent of the states plays no role in proving that the corresponding operators are contractions. So the initial Assumptions (1)-(6) hold as they are. So almost sure exponential convergence to a common value $\mathbf{1}z$, where z is the random variable described in Proposition 3, is established.

[a $\rightarrow$ b]

We define the event $A_b \in \mathcal{F}$

$$A_b := \{\omega \in \Omega : \exists z(\omega) < \infty : \lim_t \mathbf{X}_t(\omega) = \mathbf{1}z(\omega)\}.$$

and $\mathbb{P}[A_b] = 1$. Fixing $\omega \in A_b$ we have

$$\lim_t \left(\sum_i \int_0^t \sigma_i(s) dB^i(s) \right)(\omega) = Nz(\omega) + \tau \sum_{i,j} g_{ij}(z(\omega)) -$$
$$- \sum_{i,j} \int_{-\tau}^0 g_{ij}(\phi_j(s))ds - \sum_i \phi_i(0)$$

since z is almost sure finite it follows that $\lim_t \sum_i \int_0^t \sigma_i(s)dB^i(s)$ exists and it is finite a.s.. By the independence of the processes the same holds for $\lim_t \int_0^t \sigma_i(s)dB^i(s)$ separately, i.e. $\sigma_i \in L^2([0, \infty), \mathbb{R})$. ■

VI. Discussion and Concluding Remarks

We introduced a non-linear multi-agent consensus model with delays and we investigated the set of sufficient conditions so that asymptotic consensus solutions can arise. Unlike the vast majority of the related works, we used a fixed point theory argument. We applied the fundamental result of Fixed Point Theory, the Banach Contraction Mapping Principle, where the contracting map is the solution operator of our model. We assumed that the system admits a linear part whereas the remaining non-linear part behaves smoothly enough. After establishing Theorem 2, we investigated the problem of robustness by introducing deterministic but state-independent perturbations. Finally we extended the perturbations to stochastic ones in terms of Brownian motion perturbations.

A. Advantages

Using a Fixed Point argument, one needs not to worry about finding an appropriate Lyapunov function. The de facto proof of a solution in an appropriately designed complete metric space, answers simultaneously the problems of existence (in the large), uniqueness and stability. Moreover, one can use a weighted metric, like we did, to prove a stronger type of convergence. In this work, we examined exponential convergence in view of the fact that it is the type of convergence met in linear agreement dynamics.

B. Disadvantages

Undoubtedly the first and perhaps most important drawback of this approach is the number and strength of the assumptions. This point has two main directions:

1) No Monotonicity Assumptions: This is the most important factor. It is reminded that in Eq. (1) or (3), for example, a_{ij} are non-negative, whereas similar assumptions were taken in non-linear versions of this model (see (2) and [8]). Unlike these models, we did not consider any monotonicity assumptions on g_{ij}. We only took g_{ij} close enough to them. Future work should attempt to impose these types of conditions in an effort to reduce the strength of the assumptions made in this paper.

2) Non-symmetric Weights and Multiple Delays: One can consider non-symmetric weights a_{ij} at the expense of more tedious analysis and stronger assumptions. In such case the Laplacian of Eq. (3) is non-symmetric, the consensus point is $\mathbf{1}\mathbf{c}^T\mathbf{y}_0$, where $\mathbf{c}^T L = 0$ and the rate of convergence is exponential with rate $Re\{\lambda_2\} > 0$, so the bounds in Lemma 1 do not hold. Similar difficulties arise in case one would want to consider multiple or distributed delays. The method surely allows it, but the assumptions one needs to make are stronger and the delay-dependent result includes only the maximum delay (see [4] for the case of linear time invariant networks with delays).

C. State & Communication Dependent Perturbations

A final comment should be made for the perturbations considered. We remark that no intrinsically stochastic result has been invoked here, despite the necessary terminology used. This is not surprising though since the diffusion coefficients of the SDEs were state independent. Then the results of Section (V) are in fact the deterministic arguments made in Section (IV), a technique that is not uncommon (see for example [15]). A very interesting extension would be the study of this model in the presence of state dependent stochastic perturbations (which readily arise in the case of communication noise, for example). This issue is left for future work.

References

[1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[2] J. Tsitsiklis, D. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Transactions on Automatic Control*, vol. 31, no. 9, pp. 803–812, 1986.

[3] S. Motsch and E. Tadmor, "A new model for self-organized dynamics and its flocking behavior," *Journal of Statistical Physics*, vol. 144, no. 5, pp. 923–947, 2011.

[4] C. Somarakis and J. S. Baras, "A fixed point theory approach to multi-agent consensus dynamics with delays," Tech. Rep. TR-2013-01, 2013. [Online]. Available: http://hdl.handle.net/1903/13392

[5] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[6] V. Blondel, J. Hendrickx, A. Olshevsky, and J. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05. 44th IEEE Conference on*, dec. 2005, pp. 2996 – 3000.

[7] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *Automatic Control, IEEE Transactions on*, vol. 49, no. 9, pp. 1520 – 1533, sept. 2004.

[8] A. Papachristodoulou, A. Jadbabaie, and U. Munz, "Effects of delay in multi-agent consensus and oscillator synchronization," *Automatic Control, IEEE Transactions on*, vol. 55, no. 6, pp. 1471 –1477, june 2010.

[9] B. Bollobas, *Modern Graph Theory*. Springer Graduate Texts in Mathematics, 1998, vol. 184.

[10] C. Godsil and G. Royle, *Algebraic Graph Theory*. Springer, 2001, vol. 207.

[11] T. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, ser. Dover Books on Mathematics Series. Dover Publications, 2006.

[12] L. C. Evans, "An introduction to stochastic differential equations – version 1.2," UC Berkeley, Tech. Rep., 2006.

[13] H. L. Royden, *Real analysis*, 3rd ed. New York: Macmillan, 1988.

[14] B. Oksendal, *Stochastic Differential Equations*, 6th ed. Springer, 2007.

[15] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed. Springer - Graduate Texts in Mathematics, 1991.